Common fixed point of faintly compatible in fuzzy metric space

Swati Agnihotri
Department of Applied Humanities
Invertis University
Bareilly (U.P.)
India

K.K. Dubey
Department of Applied Humanities
Invertis University
Bareilly (U.P.)
India

V.K. Gupta
P.G. Department of Mathematics
Govt. Madhav Science P.G. College
Ujjain (M.P.)
India

Arihant Jain*
School of Studies in Mathematics
Vikram University
Ujjain (M.P.)
India
arihant2412@gmail.com

Abstract. Aim of this paper is to establish a common fixed point theorem for faintly compatible and subsequentially continuous self maps of fuzzy metric space and generalizing the result of Jain et al. [6].
Keywords: common fixed point, fuzzy metric space, weakly compatible mappings, occasionally weakly compatible, conditionally compatible, faintly compatible mappings, sub sequentially continuous.

1. Introduction

The concept of fuzzy set was initially investigated by Zadeh [18] as a new way to represent vagueness in everyday life. The special feature of fuzzy set is that it assign partial membership for elements in its domain, while in ordinary set theory particular element has either full membership or no membership, intermediate situation is not considered. A large number of renowned Mathematicians worked with fuzzy sets in different branches of Mathematics. One such is the Fuzzy Metric Space. In this paper, we are considering the fuzzy metric space

$*$. Corresponding author

defined by Kramosil and Michalek [9] and modified by George and Veeramani [4] with the help of continuous t-norm. In 1999,Vasuki [13] introduced the concept of R-weak commutative of mappings in fuzzy metric space.

In 1982, Sessa [14] obtained the first weaker version of commutativity by introducing the notion of weak commutativity. This concept was further generalized by Jungck [8] when he defined the concept of compatible mapping. The concept of compatibility in fuzzy metric space was proposed by Mishra et. al. [11]. In 1996, Jungck [7] again generalized the notion of compatible mapping by introducing weak mapping. Cho et. al. [3] introduced the concept of semi-compatible maps in d-topological space. Singh and Jain [15] defined the concept of semi-compatible maps in fuzzy metric space. In 2008, Al-Thagafi and Shahzad [1] generalized the notion of weak compatibility by new notion of occasionally weakly compatible (owc) mappings. Pant et. al. [12] introduced the concept of conditional compatible maps. The use of occasional weak compatibility is a redundancy for fixed point theorems under contractive conditions. To remove this redundancy we use faintly compatible mapping in our paper which is weaker than weak compatibility or semi compatibility. Faintly compatible maps introduced by Bisht and Shahzad [2] as an improvement of conditionally compatible maps.

In 2007, Singh et. al. [17] proved common fixed point theorem using the concept of compatible and weak compatible in fuzzy metric space. Subsequently, in 2014, Jain et. al. [6] established fixed point theorem for six self maps by using concept of occasionally weak compatible maps and generalized the result of Singh et. al. [17]. Jain et. al. [5] introduced the notion of subsequential continuous mappings in fuzzy metric space which is more general than continuous mappings as well as reciprocal continuous mappings and also introduced the concept of occasionally weakly compatible mappings which is more general than weakly compatible mappings in fixed point theory in 2014.

In this paper, we generalize the result of Jain et. al. [6] by replacing the occasionally weakly compatible maps to faintly subsequential continuous maps.

2. Preliminaries

Definition 2.1 ([18]). Let X be any set. A fuzzy set A in X is a function with domain in X and values in [0, 1].

Definition 2.2 ([11]). A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is called a continuous t-norm if it satisfies the following conditions:

(i) $*$ is associative and commutative,
(ii) $*$ is continuous,
(iii) $a * 1 = a$, for all $a \in [0,1]$,
(iv) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, for all $a, b, c, d \in [0,1]$.

Examples of t-norms are

$a * b = \min\{a, b\}$ (minimum t-norm),
$a * b = ab$ (product t-norm).

Definition 2.3 ([11])**.** The 3-tuple (X,M,*) is called a fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times (0, \infty)$ satisfying the following conditions:

(FM-1) $M(x, y, t) > 0$,
(FM-2) $M(x, y, t) = 1$ if and only if $x = y$,
(FM-3) $M(x, y, t) = M(y, x, t)$,
(FM-4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,
(FM-5) $M(x, y, .) : (0, \infty) \to [0, 1]$ is continuous, ; for all $x, y, z \in X$ and $t, s > 0$.

Let (X, d) be a metric space and let $a * b = ab$ or $a * b = \min\{a, b\}$ for all $a, b \in [0, 1]$.

Let $M(x, y, t) = \frac{t}{t+d(x,y)}$; for all $x, y \in X$ and $t > 0$.

Then (X,M,*) is a fuzzy metric space, and this fuzzy metric M induced by d is called the standard fuzzy metric [11].

Definition 2.4 ([11])**.** A sequence $\{x_n\}$ in a fuzzy metric space (X,M,*) is said to be convergent to a point $x \in X$,if $M(x_n, x, t) = 1$ for all $t > 0$.

Further, the sequence $\{x_n\}$ is said to be Cauchy if $M(x_n, x_{n+p}, t) = 1$, for all $t > 0$ and $p > 0$.

The space (X,M,*) is said to be complete if every Cauchy sequence in X is convergent in X.

Lemma 2.5 ([10])**.** Let (X,M,*) be a fuzzy metric space. Then M is non-decreasing for all $x, y \in X$.

Lemma 2.6 ([11])**.** Let (X,M,*) be a fuzzy metric space. Then M is a continuous function on $X^2 \times (0, \infty)$.

Throughout this paper (X,M,*) will denote the fuzzy metric space with the following condition:

(FM-6) $\lim_{n\to\infty} M(x, y, t) = 1$ for all $x, y \in X$ and $t > 0$.

Definition 2.7 ([16])**.** Let $f$and g be self mappings on a fuzzy metric space (X,M,*).

The pair (f, g) is said to compatible if

$$\lim_{n\to\infty} M(fgx_n, gfx_n, t) = 1$$

for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z, \text{ for some } z \in X.$$

Definition 2.8 ([17])**.** Let f and g be self mappings on a fuzzy metric space (X,M,*). Then the mappings are said to be weakly compatible if they commute at their coincidence points, that is, $fx = gx$ implies $fgx = gfx$.

It is known that a pair of (f, g) compatible maps is weakly compatible but converse is not true in general.

Definition 2.9 ([15])**.** A pair (A, B) of self maps of a fuzzy metric space (X,M,*) is said to be semi-compatible if $\lim_{n\to\infty} ABx_n = Bx$, whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = x.$$

Definition 2.10 ([6])**.** Self maps A and S of a Fuzzy metric space (X,M,*) are said to be occasionally weakly compatible (owc) if and only if there is a point x in X which is a coincidence point of A and S at which A and S commute.

It follows that if (A, B) is semi-compatible and $Ax = Bx$ then $ABx = BAx$ that means every semi-compatible pair of self maps is weak compatible but the converse is not true in general.

Definition 2.11 ([5])**.** Two self maps A and S on a fuzzy metric space are called reciprocal continuous if $\lim_{n\to\infty} ASx_n = At$ and $\lim_{n\to\infty} SAx_n = St$ for some t in X whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t.$$

Definition 2.12 ([5])**.** Two self maps A and S on a fuzzy metric space are said to be sub sequentially continuous if and only if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t \text{ for some in } X \text{ and satisfy}$$

$$\lim_{n\to\infty} ASx_n = At \text{ and } \lim_{n\to\infty} SAx_n = St.$$

Clearly, if A and S are continuous then they are obviously sub-sequentially continuous. The next example shows that there exist sub-sequential continuous pairs of mappings which are neither continuous nor reciprocally continuous.

Example 2.13. Let $X = R$, endowed with metric d and $M_d(x, y, t) = \frac{t}{t+d(x,y)}$ for all $x, y \in X$, all $t > 0$ define the self mappings A, S as follow

$$A(x) = \left\{ \begin{array}{ll} 2, & x < 3 \\ x, & x \geq 3 \end{array} \right\} \text{and } S(x) = \left\{ \begin{array}{rl} 2x - 4, & x \leq 3 \\ 3, & x > 3 \end{array} \right\}.$$

Consider a sequence $x_n = 3 + \frac{1}{n}$; then,

$$A(x_n) = \left(3+\frac{1}{n}\right) \to 3, S(x_n) = 3,\ SA(x_n) = S\left(3+\frac{1}{n}\right) = 3 \neq S(3) = 2, \text{as } n \to \infty.$$

Thus A and S are not reciprocally continuous but, if we consider a sequence $x_n = 3 - \frac{1}{n}$, then,

$$A(x_n) = 2,\ S(x_n) = 2\Big(3 - \frac{1}{n}\Big) - 4 = \Big(2 - \frac{2}{n}\Big) = 2 \text{ as } n \to \infty$$
$$AS(x_n) = A\Big(2 - \frac{2}{n}\Big) = 2 = A(2),\ SA(x_n) = S(2) = 0 = S(2) \text{ as } n \to \infty.$$

Therefore, A and S are sub sequentially continuous.

Remark 2.14 ([5])**.** If A and S are continuous or reciprocally continuous then they are obviously sub sequentially continuous, but converse is not true.

Definition 2.15 ([2])**.** Two self maps A and S on a fuzzy metric space are said to be conditionally compatible if and only if whenever the set of sequences y_n satisfying $\lim_{n\to\infty} A(y_n) = \lim_{n\to\infty} S(y_n)$ is nonempty, there exists a sequence z_n such that $\lim_{n\to\infty} A(z_n) = \lim_{n\to\infty} S(z_n) = u$ and

$$\lim_{n\to\infty} M\big(A(S(z_n)), S(A(z_n)), t\big) = 1.$$

Definition 2.16 ([2])**.** Two self-mappings A and S of a metric space (X, d) will be called faintly compatible iff A and S are conditionally compatible and A and S commute on a nonempty subset of coincidence points whenever the set of coincidences is nonempty.

If A and S are compatible, then they are obviously faintly compatible, but the converse is not true in general.

Example 2.17. Let $X = [3, 6]$ and d be the usual metric on X. Define self-mappings A and S on X as follows:

$$A(x) = 3 \text{ if } x = 3 \text{ or } x > 5,\ A(x) = x + 1 \text{ if } 3 < x \leq 5.$$
$$S(3) = 3,\ S(x) = \frac{(x+7)}{3} \text{ if } 3 < x \leq 5,\ S(x) = \frac{(x+1)}{2} \text{ if } x > 5.$$

In this example A and S are faintly compatible but not compatible.

To see this, if we consider the constant sequence $\{y_n = 3\}$, then A and S are faintly compatible.

On the other hand, if we choose the sequence $\Big\{x_n = 5 + \frac{1}{n}\Big\}$, then

$$\lim_{n\to\infty} A(x_n) = 3 = \lim_{n\to\infty} S(x_n) \text{ and } \lim_{n\to\infty} M\Big(A(S(x_n)), S(A(x_n)), t\Big) \neq 0.$$

Thus A and S are faintly compatible, but they are not compatible.

In 2014, Jain et. al. [6] proved the following result:

Theorem 2.18. Let A, B, S and T be self mappings of a complete Fuzzy metric space (X,M,*). Suppose that they satisfy the following conditions:

(2.15.1) $A(X) \subseteq T(X),\ B(X) \subseteq S(X)$;
(2.15.2) The pairs (A, S) and (B, T) are occasionally weakly compatible,
(2.15.3) There exists $k \in (0, 1)$ such that $\forall\ x, y \in X$ and $t > 0$,

$$M(Ax, By, kt) \geq Min\big\{M(By, Ty, t), M(Sx, Ty, t), M(Ax, Sx, t)\big\}.$$

Then A, B, S and T have a unique common fixed point in X.

3. Main result

Theorem 3.1. Let A, B, P, Q, S and T be self-mappings of fuzzy metric space (X,M,*). Suppose that they satisfy the following condition:

[3.1.1] $A(X) \subseteq QT(X)$ and $B(X) \subseteq PS(X)$,
[3.1.2] (A, PS) and (B, QT) are faintly compatible and subsequently continuous,
[3.1.3] $AS = SA, BT = TB, QT = TQ$ and $PS = SP$.
[3.1.4] Their exist $k \in (0, 1)$ such that $\forall\, x, y \in X$ and $t > 0$

$$M(Ax, By, kt) \geq \min\{M(By, QTy, t), M(PSx, QTy, t), M(Ax, PSx, t)\}.$$

Then A, B, P, Q, S and T have a unique common fixed point in X.

Proof. Let $x_0 \in X$ be an arbitrary point.

From condition [3.1.1], $A(X) \subseteq QT(X)$ and $B(X) \subseteq PS(X)$

There exist x_1 and $x_2 \in X$ such that

$$A(x_0) = QT(x_1) \text{ and } B(x_1) = PS(x_2)$$

We can construct sequences $\{y_n\}$ and $\{x_n\}$ in X such that

$$\begin{aligned} y_{2n} &= A(x_{2n}) = QT(x_{2n+1}) \\ y_{2n+1} &= B(x_{2n+1}) = PS(x_{2n+2}) \text{ for } n = 0, 1, 2, 3, ... \end{aligned}$$

We show that $\{y_n\}$ is a Cauchy sequence in X.

Using equation (3.1.4) with $x = x_{2n}, y = x_{2n+1}$,

$$\begin{aligned} M(Ax_{2n}, Bx_{2n+1}, kt) &= M(y_{2n}, y_{2n+1}, kt) \\ &\geq \min\{M(Bx_{2n+1}, QTx_{2n+1}, t), M(PSx_{2n}, QTx_{2n+1}, t), \\ &\quad M(Ax_{2n}, PSx_{2n}, t)\} \end{aligned}$$

$$\begin{aligned} M(y_{2n}, y_{2n+1}, kt) &\geq \min\{M(y_{2n+1}, y_{2n}, t), M(y_{2n-1}, y_{2n}, t), M(y_{2n}, y_{2n-1}, t)\} \\ &\geq \min\{M(y_{2n+1}, y_{2n}, t), M(y_{2n}, y_{2n-1}, t)\} \end{aligned}$$

Thus, we have

$$
\begin{aligned}
M(y_{2n}, y_{2n+1}, t) &\geq \min\{M(y_{2n+1}, y_{2n}, t/k), M(y_{2n}, y_{2n-1}, t/k)\} \\
M(y_{2n}, y_{2n+1}, kt) &\geq \min\{M(y_{2n+1}, y_{2n}, t/k), M(y_{2n}, y_{2n-1}, t/k), \\
&\qquad M(y_{2n}, y_{2n-1}, t)\} \\
M(y_{2n}, y_{2n+1}, kt) &\geq \min\{M(y_{2n+1}, y_{2n}, t/k), M(y_{2n}, y_{2n-1}, t)\} \\
&\geq \min\{M(y_{2n+1}, y_{2n}, t/k^2), M(y_{2n}, y_{2n-1}, t/k^2), \\
&\qquad M(y_{2n}, y_{2n-1}, t)\} \\
&\geq \min\{M(y_{2n+1}, y_{2n}, t/k^2), M(y_{2n}, y_{2n-1}, t)\} \\
&\geq \min\{M(y_{2n+1}, y_{2n}, t/k^m), M(y_{2n}, y_{2n-1}, t)\}.
\end{aligned}
$$

Taking limit as $m \to \infty$

$$M(y_{2n}, y_{2n+1}, kt) \geq M(y_{2n}, y_{2n-1}, t);\ \forall\ t > 0.$$

Similarly

$$M(y_{2n+1}, y_{2n+2}, kt) \geq M(y_{2n+1}, y_{2n}, t);\ \forall\ t > 0.$$

Thus, for all n and $t > 0$

$$M(y_n, y_{n+1}, kt) \geq M(y_n, y_{n-1}, t).$$

Therefore,

$$M(y_n, y_{n+1}, t) \geq M(y_{n-1}, y_n, t/k) \geq M(y_{n-2}, y_{n-1}, t/k^2) \geq ... \geq M(y_0, y_1, t/k^n).$$

Hence, $\lim_{n\to\infty} M(y_n, y_{n+1}, t) = 1;\ \forall\ t > 0$.

Now, for any integer p, we have

$$M(y_n, y_{n+p}, t) \geq M(y_n, y_{n+1}, t/p) * M(y_{n+1}, y_{n+2}, t/p) * ... * M(y_{n+p-1}, y_{n+p}, t/p).$$

Therefore,

$$
\begin{aligned}
&\lim_{n\to\infty} M(y_n, y_{n+p}, t) = 1 * 1 * 1 * ... * 1 = 1 \\
&\lim_{n\to\infty} M(y_n, y_{n+p}, t) = 1.
\end{aligned}
$$

This show that $\{y_n\}$ is Cauchy sequence in X which is complete therefore $\{y_n\}$ converges to $u \in X$.

Then, subsequences $\{A(x_{2n})\}, \{B(x_{2n+1})\}, \{QT(x_{2n+1})\}$ and $\{PS(x_{2n+2})\}$ are also converges to $u \in X$.

$$\lim_{n\to\infty} A(x_{2n}) = \lim_{n\to\infty} PS(x_{2n}) = \lim_{n\to\infty} B(x_{2n+1}) = \lim_{n\to\infty} QT(x_{2n+1}) = u.$$

Case (1): (A, PS) is faintly compatible and sub sequentially continuous.

$\lim_{n\to\infty} A(x_{2n}) = \lim_{n\to\infty} PS(x_{2n}) = u$ and (A, PS) is faintly compatible then, there exist sequence $\{z_n\}$ in X, where, $\lim_{n\to\infty} A(z_n) = \lim_{n\to\infty} PS(z_n) = v$ for some $v \in X$ such that

$$\lim_{n\to\infty} M(PSA(z_n), APS(z_n), t) = 1.$$

As (A, PS) sub sequentially continuous, we have

$$\lim_{n\to\infty} Az_n = v \Rightarrow \lim_{n\to\infty} PS(Az_n) = PSv,$$

and

$$\lim_{n\to\infty} PSz_n = v \Rightarrow \lim_{n\to\infty} A(PSz_n) = Av,$$

Since,

$$\lim_{n\to\infty} M(PSAz_n, APSz_n, t) = 1,$$

$$PSv = Av. \tag{1}$$

Case (2): (B, QT) is faintly compatible and subsequently continuous. We know that $\lim_{n\to\infty} B(x_{2n+1}) = \lim_{n\to\infty} QT(x_{2n+1}) = u$ and (B, QT) is faintly compatible then there exist sequence $\{z'_n\}$ in X
where, $\lim_{n\to\infty} B(z'_n) = \lim_{n\to\infty} QT(z'_n) = v'$ for some $v \in X$ such that

$$\lim_{n\to\infty} M\Big(B(QTz'_n), QT(Bz'_n), t\Big) = 1.$$

As (B, QT) subsequently continuous we have

$$\lim_{n\to\infty} Bz'_n = v' \Rightarrow \lim_{n\to\infty} QT(Bz'_n) = QTv'$$
$$\lim_{n\to\infty} QTz'_n = v' \Rightarrow \lim_{n\to\infty} B(QTz'_n) = Bv'$$

Since,

$$\lim_{n\to\infty} M(BQTz'_n, QBTz'_n, t) = 1$$

$$Bv' = QTv'. \tag{2}$$

Since pairs (A, PS) and (B, QT) are faintly compatible, we have

$$Av = PSv$$

$$\Rightarrow AAv = APSv = PSAv = PS(PSv) \tag{3}$$

and $Bv' = QTv'$

$$\Rightarrow BBv' = BQTv' = QTBv' = QT(QTv\prime). \tag{4}$$

Now, we show that $Av = Bv'$

Using inequality [3.1.4] with $x = v$ and $y = v'$,

$$\begin{aligned}
M(Ax, By, kt) &\geq \min\{M(By, QTy, t), M(PSx, QTy, t), M(Ax, PSx, t)\} \\
M(Av, Bv', kt) &\geq \min\{M(Bv', QTv', t), M(PSv, QTv', t), M(Av, PSv, t)\} \\
&\geq \min\{M(Bv', Bv, t), M(Av, Bv', t), M(Av, Av, t)\} \\
&\geq \min\{1, M(Av, Bv', t), 1\} \quad \text{by using (1) and (2)} \\
M(Av, Bv', kt) &\geq M(Av, Bv', t).
\end{aligned}$$

$$Av = Bv'. \tag{5}$$

Now we show that $A(Av) = Av$.

Using equation [3.1.4] with $x = Av$ and $y = v'$,

$$\begin{aligned}
M(AAv, Bv', kt) &\geq \min\{M(Bv', QTv', t), M(PSAv, QTv', t), M(AAv, PSAv, t)\} \\
M(AAv, Av, kt) &\geq \min\{M(Bv', Bv', t), M(AAv, Bv', t), M(AAv, AAv, t)\} \\
M(AAv, Av, kt) &\geq \min\{1, M(AAv, Av, t), 1\} \\
M(AAv, Av, kt) &\geq M(AAv, Av, t)
\end{aligned}$$

$$AAv = Av. \tag{6}$$

Therefore, Av is fixed point of mapping A.

Again, we show that

$$B(Av) = (Av) \quad \text{or} \quad BBv' = Av.$$

Putting $x = v$ and $y = Bv'$ in [3.1.4]

$$M(Av, BBv', kt) \geq \min\{M(BBv', QTBv', t), M(PSv, QTBv', t), M(Av, PSv, t)\}.$$

Using (1) & (4),

$$\begin{aligned}
M(Av, BBv', kt) &\geq \min\{M(BBv', BBv', t), M(Av, BBv', t), M(Av, Av, t)\} \\
M(Av, BBv', kt) &\geq \min\{1, M(Av, BBv', t), 1\} \\
M(Av, BBv', kt) &\geq M(Av, BBv', t) \\
M(Av, BBv', kt) &= 1 \\
Av &= BBv'
\end{aligned}$$

$$\text{or} \qquad B(Av) = Av. \tag{7}$$

Therefore, Av is fixed point of mapping B.

Using equations (3), (4), (5) and (6)

$$A(Av) = B(Av) = PS(Av) = QT(Av) = Av. \tag{8}$$

Putting $x = Sv$ and $y = v'$ in inequality [3.1.4]

$$\begin{aligned}
M(ASv, Bv', kt) &\geq \min\{M(Bv', QTv', t), M(PS(Sv), QTv', t),\\
&\qquad M(ASv, PSSv, t)\}\\
M(SAv, Bv', kt) &\geq \min\{M(Bv', Bv', t), M(SPSv, Bv', t), M(ASv, S(PSv), t)\}\\
M(SAv, Bv, kt) &\geq \min\{1, M(SAv, Av, t), M(ASv, SAv, t)\}\\
&\geq \min\{1, M(SAv, Av, t), 1\}\\
M(SAv, Bv', kt) &\geq M(SAv, Av, t)
\end{aligned}$$

$$SAv = Av \tag{9}$$

$PS(Av) = Av$ by using equation (8)

$$P(Av) = Av \tag{10}$$

Av is also fixed point of mappings P and S.

Therefore,

$$A(Av) = B(Av) = P(Av) = S(Av) = Av.$$

Now, using equation [3.1.4] with $x = v$ and $y = Tv'$,

$$\begin{aligned}
M(Av, BTv', kt) &\geq \min\{M(BTv', QT(Tv'), t), M(PSv, QT(Tv'), t),\\
&\qquad M(Av, PSv, t)\}\\
M(Av, TBv', kt) &\geq \min\{M(BTv', TQ(Tv'), t), M(PSv, TQ(Tv'), t),\\
&\qquad M(Av, Av, t)\}\\
M(Av, TAv, kt) &\geq \min\{M(BTv', TBv', t), M(Av, TBv', t),\\
&\qquad M(Av, Av, t)\}, \text{ since } (2)\\
&\geq \min\{M(BTv', BTv', t), M(Av, TAv, t), 1\}\\
&\geq \min\{1, M(Av, TAv, t), 1\}\\
M(Av, TAv, kt) &\geq M(Av, TAv, t)\\
M(Av, TAv, kt) &= 1
\end{aligned}$$

$$Av = TAv. \tag{11}$$

Therefore, Av is fixed point of mapping T.

Using equation (8) and (11)

$$QT(Av) = Av$$

$$Q(Av) = Av. \tag{12}$$

Av is also fixed point of mapping Q

We get that there is a point Av in set X such that

$$A(Av) = B(Av) = S(Av) = P(Av) = T(Av) = Q(Av) = Av.$$

Av is a common fixed point of mappings A, B, S, P, T and Q in X.

Uniqueness. Let v and w are two common fixed points of mappings A, B, S, P, T and Q. Then,

$$Av = Bv = Sv = Pv = Tv = Qv = v \tag{13}$$

$$\text{and} \qquad Aw = Bw = Sw = Pw = Tw = Qw = w. \tag{14}$$

Now we have to show that $v = w$.

Putting $x = v$ and $y = w$ in inequality [3.1.4],

$$\begin{aligned} M(Av, Bw, kt) &\geq \min\{M(Bw, QTw, t), M(PSv, QTw, t), M(Av, PSv, t)\} \\ M(v, w, kt) &\geq \min\{M(w, Qw, t), M(Pv, Qw, t), M(v, PSv, t)\} \\ M(v, w, kt) &\geq \min\{1, M(v, w, t), 1\} \\ M(v, w, kt) &\geq M(v, w, t). \end{aligned}$$

$v = w$. If we take $S = T = I$ the identity mappings on X in theorem 3.1.

Corollary 3.2. Let A, B, P and Q be self mappings of complete fuzzy metric space (X,M,*). Suppose that they satisfy the following conditions:

[3.1.1] $A(X) \subseteq Q(X)$ and $B(X) \subseteq P(X)$,

[3.1.2] (A, P) and (B, Q) are faintly compatible and subsequently continuous,

[3.1.3] Then exist $k \in (0, 1)$ such that $\forall\ x, y \in X$ and $t > 0$,

$$M(Ax, By, kt) \geq \min\{M(By, Qy, t), M(Px, Qy, t), M(Ax, Px, t)\}.$$

then A, B, P and Q have a unique common fixed point in X.

Proof. The proof is similar to the proof of theorem (3.1).

Corollary 3.3. Let A, P and Q be self mappings of complete fuzzy metric space (X,M,*) satisfy the following conditions:

[3.1.1] $A(X) \subseteq P(X) \cap Q(X)$.

[3.1.2] (A, P) and (A, Q) are faintly compatible and subsequently continuous,

[3.1.3] $M(Ax, Ay, kt) \geq \min\{M(Ay, Qy, t), M(Px, Qy, t), M(Ax, Px, t)\}$, for all $x, y \in X,\ t > 0$ and $k \in (0, 1)$.

Then A, P and Q have a unique common fixed point in X.

If X is not complete and (3.2.1) may or may not be satisfy for these four self mappings.

Corollary 3.4. Let A, B, P and Q be self mappings of fuzzy metric space (X,M,*). Suppose that they satisfy the following conditions:

[3.4.1] $M(Ax, By, kt) \geq \min\{M(By, Qy, t), M(Px, Qy, t), M(Ax, Px, t)\}$;
for all $x, y \in X,\ t > 0$ and $k \in (0, 1)$

[3.4.2] If pairs (A, P) and (B, Q) are non-compatible faintly compatible and subsequently continuous.

Then, A, B, P and Q mappings have a unique common fixed point in X.

Proof. (A, P) and (B, Q) are non-compatible then there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that $A(x_n) = P(x_n) = u$, for some $u \in X$. But $M(APx_n, PAx_n, t) \neq 1$ and $\lim_{n\to\infty} B(y_n) = \lim_{n\to\infty} Q(y_n) = u'$ for some $u' \in X$,

But $M(By_n, Qy_n, t) \neq 1$.

(A, P) and (B, Q) are non-compatible faintly compatible, so it implies that

$$\lim_{n\to\infty} A(x_n) = \lim_{n\to\infty} P(x_n) = u \text{ and } (A, P)$$

is faintly compatible subsequentially continuous

$$\text{and } \lim_{n\to\infty} B(y_n) = \lim_{n\to\infty} Q(y_n) = u' \text{ and } (B, Q)$$

is faintly compatible subsequentially continuous

Rest proof is similar to case (I) and case (II).

4. Conclusion

Our result is a generalization of the result of Jain et.al. [6] in the sense that we have replaced occasionally weakly compatible (owc) to faintly compatible and prove a theorem on common fixed point theorems for six self mappings in complete fuzzy metric space. Corollary 3.4 is also another generalization of Jain et.al. [6] where completeness is not necessary.

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